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Developing a Sense of Scale: Looking Backward

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Abstract: Although scale has been identified as one of four major interdisciplinary themes that cut across the science domains by the American Association for the Advancement of Science (1989), we are only beginning to understand how students learn and apply scale concepts. Early research on learning scale tended to focus on perceptions of linear distances, navigation and way finding, whereas more recent work has examined how students conceptualize scale in science contexts. This study sought to understand how understandings of scale develop from childhood to adulthood by asking 50 professionals to reflect on their experiences (in and out-of-school) learning about scale. Semi-structured interviews were utilized to obtain information about educational experiences, informal experiences, and applications of scale in different professions. Results showed that most of the participants used anchor points as conceptual benchmarks when applying scale in their job. Seventy-six percent of the participants attributed physical experiences such as moving through the environment by car, walking, bicycling, or flying in an airplane as contributing to the development of a sense of scale. Results of this study were used to develop a possible model of the trajectory of scale concepts that develop as individuals move from novice through increasing degrees of expertise. Across professions, participants emphasized the critical role that scale plays in their work. For many, scale was viewed as central to accomplishing the work-related tasks. © 2009 Wiley Periodicals, Inc. J Res Sci Teach 46: 460–475, 2009

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Introduction

Scale is one of four major interdisciplinary themes that have been identified by the American Association for the Advancement of Science's *Benchmarks for Science Literacy* that cut across the science domains (American Association for the Advancement of Science, 1989). We are only beginning to understand how students develop concepts of scale and how they apply scale across different science contexts. This study describes the reflections of 50 professionals from a wide range of fields as they looked back on their in and out-of-school experiences learning about scale while growing up. The applications of scale in different types of jobs are also discussed.

Understanding and applying scale involves a number of concepts and processes such as quantity, distance, measurement, estimation, proportion, and perspective. Although most applications of scale involve linear distances, other variables such as temperature, time, volume, or mass are also important. Lock and Molyneaux (2006) write, ''Scale is a slippery concept, one that is sometimes easy to define but often difficult to grasp. . . there is much equivocation about scale, as it is at the same time a concept, a lived experience and an analytical framework'' (p. 1). In this paper we define scale broadly to include any quantification of a property that is measured. Although most of the applications examined in this study are linear measures, other less well-known measures (such as scales for wind or light intensity) are also examined.

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Some of the most significant new advancements in science are being made at the extreme ends of science scales (the very small and the very large). At the large end of scale, researchers are asking questions about the size and origin of the universe whereas at the very small end of the scale, nanoscale scientists are probing the behavior of materials in a world that exists at a billionth of a meter. If we are to help students who will become the next generation of scientists work across many orders of magnitude, we need to better understand how they learn scale in and out-of-school.

Previous Research on Scale

Early research on learning scale focused on its application in geography as well as perceptions of linear distances. These original studies tended to focus on navigation and wayfinding (e.g., Golledge, Gale, Pellegrino, & Doherty, 1992; Hegarty, Montello, Richardson, Ishikawa, & Lovelace, 2006). More recent work has examined how students conceptualize scale in science contexts. Trend (2001) studied conceptions of deep geological time and found that an individual's conception of time tends to be more detailed nearer to human time scales and more broadly conceptualized at the distant scales. Trend notes that for most people there are three clusters of time that include less ancient, moderately ancient, and extremely ancient.

Hegarty et al. (2006) also reported three conceptual divisions for spaces that included environmental, vista and figural. Environmental space is that scale of space that allows the individual to move through it via locomotion. Vista space is space that is external to the body and is seen without moving such as looking at a sunset or a mountain range. Figural space is defined by Hegarty et al. as small space that is seen only from one vantage point.

One of the few studies of students' concepts of scale was done by Tretter, Jones, Andre, Negishi, and Minogue (2006) who examined elementary, middle, high school, and graduate students' concepts of scale and found that students tend to hold distinct conceptual categories of size and that these categories of size differed across the different ages and levels of schooling. This study also reported that students tended to be more accurate at understanding relative size than absolute size. In addition, when students could experience a size or scale kinesthetically (in human scale ranges) they were more accurate in their understandings.

In another study, Tretter, Jones, and Minogue (2006) examined the accuracy of spatial scale conceptions of science phenomena across 215 participants that included students in 5th grade, 7th grade, 9th grade, 12th grade, and doctoral students in science. Conceptions from 1 nm to 1 billion meters were explored through use of written assessments, a card sort activity, and individual interviews. Results showed that the accuracy of small scale was not symmetrical to the accuracy of large scale. Large-scale accuracy tended to decline uniformly as the scale was further removed from human-scale, whereas small-scale accuracy was discontinuous at the microscopic level. In order to conceptually interact with scales far from human scales, doctoral students in this study reported using strategies of mentally jumping to a new scale-world and creating different units of measure appropriate for that new world. Participants at the graduate level tended to use mathematics as the tool to transition between different scales.

One of the challenges in understanding large and small scale is the need for students to understand huge magnitudes of differences in scales. Jones et al. (2007) measured students' proportional reasoning and then examined students' concepts of scale before and after watching the video the *Powers of Ten* (Eames Office, 2008). The results of this study showed that accuracy in scale concepts was correlated with proportional reasoning and could be improved through watching the video. Other studies have also shown the importance of proportional reasoning in learning scale concepts.

Proportional reasoning has also been shown to be correlated with the ability to estimate linear distances (Jones, Taylor, & Broadwell, in press). In this study, middle school students' estimation of linear distances was examined for small scale (mm) to large scale (100 m) and changes in knowledge from pre- to post-instruction were measured. Results showed that teaching students to use rough body measures as tools (a body ruler) for estimation had a significant influence on their estimation accuracy. After instruction, students were better able to estimate the sizes of objects, use their body in making estimations of size, and estimate while touching an object or pacing a distance. Furthermore, proportional reasoning was significantly correlated with students' post-instruction achievement.

Taylor and Jones (in press) found a significant correlation between proportional reasoning ability and middle school students' ability to understand surface area to volume relationships. In a subsequent study, Taylor and Jones (2008) reported that there were significant correlations among proportional reasoning, visual spatial ability, and the ability to understand surface area to volume relationships not only for middle school students but also for high school students and teachers. Regression results indicated that participants' proportional reasoning and visual-spatial scores could be a possible predictor for one's ability to understand surface area to volume relationships. This study showed that learning and applying scale, specifically understanding surface area to volume relationships, is dependent on students' abilities to apply proportional reasoning and visual-spatial skills.

Experienced and novice teachers' accuracy in using scale were examined in a study by Jones, Tretter, Taylor, and Oppewal (2008) and the results showed that both groups of teachers were most accurate at the human (macroscale) scale. At small scales (below a millimeter), teachers in both groups tended to hold concepts of size that were too large and at large scales (above a meter) they tended to hold concepts of size that were too small.

Dawkins (2006), the evolutionary biologist, has argued that humans have evolved to operate in the human-scaled world. He noted, "Our brains have evolved to help us survive within the orders of magnitude of size and speed which our bodies operate at. We never evolved to navigate in the world of atoms. If we had, our brains would probably perceive rocks as full of empty space (TED Talks, retrieved April 4, 2008 from, http://www.ted.com/index.php/talks/view/id/98). Other researchers have also argued that we are uniquely suited and perhaps limited to perceiving the macroworld. Lock and Molyneaux suggest that "(s)ince our physical and conceptual systems have developed at this bodily scale, there is no guarantee that we can understand relationships in the larger and smaller worlds that these days technology allows us to visit or construct" (pp. 1–20). But if Lock and Molyneaux were correct we would not have the great advances that have taken place at the many unseen scales where science is pushing the boundaries. The tools such as the telescope and the microscope that Lock and Molyneaux refer to have opened immense new worlds that humans have not only perceived, but furthermore, they have developed the skills to explore and operate in these unseen worlds. By looking closely at how understandings of scale develop from childhood to adulthood we can better educate our citizens and future scientists to understand scientific advancements that take place at these extremes of scale.

Development of Knowledge and Skills

Research on the development of knowledge by professionals can shed light on how concepts and skills develop over time. By looking closely at those who use scale routinely in their jobs, we can begin to understand the conceptualization and problem solving strategies that are used in different domains (Chi, Feltovich, & Glaser, 1981). The early studies of expertise focused on expert chess players (Chase and Simon, 1973) and showed that expert players had superior memory of chess games to the novice players. Furthermore, this study and others have shown that experts tend to organize information in chunks that can be easily retrieved (Chase and Simon, 1973). It has also been shown that novices tend to organize information superficially whereas experts tend to use broader, more general underlying principles in organizing and approaching problems. However, the single factor that seems to contribute consistently to the development of expertise is repeated experience and practice (Charness, Krampe, & Mahr, 1996),

Study Goals

In the present study we sought to understand how expertise in scale develops from childhood through adulthood by asking 50 professionals to reflect on their experiences and knowledge with scale. This retrospective look sought to understand which in and out-of-school experiences learning about scale were remembered as contributing to their ability to use and apply scale. Understanding how these experiences promote an understanding of scale can contribute to the development of a learning trajectory that can inform the sequencing of scale instruction and the design of curricula. A further goal of this study was to document the types of scale used across different work contexts and domains.

Research Questions

1. What types of scales and scaling strategies are used by professionals in different work domains?
2. What types of formal (in-school) experiences contribute to an individual's perceived skill in using scale?

3. What types of informal (out-of-school) experiences contribute to an individual's perceived skill in using scale?
4. How do professionals perceive the role of scale in their work?

Methodology

Participants

Broadly, we sought to understand how scale and scaling was learned and how people in the community apply scale in real-world settings. Study participants were part of an ongoing series of studies that are examining concepts of size and scale. Participants were targeted and selected for participation based on the type of profession in which they were employed. A list of professions that tend to use scale in the everyday workplace was generated and participants were sought from these professions to include a wide range of training and educational preparation including those with technical skills (construction, muralist, and chef) as well as those with doctoral degrees (neurologist, materials scientist, astrophysicist). Nobel laureates and members of the National Academy of Sciences who were in scale-related areas were also invited to participate as exemplars of those with expert knowledge. Their domain-based expertise has been validated by external committees as part of the award selection process. See Table 1 for a description of participants.

The selection process began with the generation of the list of desired professions that used scale. Each of the major disciplines of science was included with emphasis given to those who worked in very large (astrophysics) and very small (nanoscale researchers) areas of science. Additional professions that utilize scale were added to the list of targeted professions (such as muralist and epidemiologist). Nominations were sought from colleagues through email requests and word of mouth for names of potential study participants who worked in the scale-related professions and were located within driving distance of the researchers. All of the participants nominated were employed in a field that required them to use scale, were considered by the nominating colleague as successful in their specific field, and had at least 5 years of experience in their current job. Because the goal was to document how scale and scaling is learned and applied across different fields (a wide view of the development of knowledge and skills), we did not attempt to control for individual differences such as age, years of experience, or backgrounds. We sought participants who were of different genders, race, and cultural backgrounds but were limited by the lack of diversity found in many of the fields (e.g., physics, materials science, engineering, etc.). There were 9 females and 41 males (45 Caucasians, 2 Hispanics, 2 African Americans, and 1 Asian). Participants were employed in a variety of different contexts including businesses (36%), industry (12%), academia (40%), local and state government (6%), federal government and agencies (6%). The majority (70%) of the participants had completed a masters or doctorate degree, 28% had an undergraduate degree, and 2% had completed high school.

Procedures

Participants were contacted by phone or email and invited to participate in a study of scale and scaling in different professions with the goal of understanding how people develop understandings of the sizes of things and scale. Individual interviews were scheduled at participants' convenience. Interviews lasted approximately 1 hour and followed a semi-structured interview protocol that sought information about educational experiences, out-of-school (informal) experiences, and applications of scale in different professions (see the interview protocol in Appendix). Participants' responses were probed for each question until the individual indicated that they had nothing more to add to their response.

Analyses

Interviews were audio recorded and transcribed for analysis. Grounded theory (Glaser, 1995) was used to derive the categories and levels of analyses. In the level 1 analysis, the transcriptions were initially qualitatively coded for frequency of responses by question by four researchers. For example, responses to the question that asked if there were experiences out of school that have contributed to the participant's sense of scale were coded as either yes or no and the examples given were tallied for frequency of responses. A series of level 2 codes were derived from the first reading and these included: in-school experiences, out-of-school experiences, uses of scale in the job, types of scale, and changes in scales. Transcripts were coded with the

Table 1
Work domain, importance of scale, applications, and tools of scale

Domain	Importance of scale	Applications and types of scale	Tools
Physical sciences			
Chemist	Integral	Mass, volumes, solution prep and precision	Glassware, pipettes
Chemist	It's key. I mean in chemistry it is key. Again because we are living in a microscopic world and all the things that are composed are microscopic	Chemistry is nano	Volumetric flasks, graduated cylinders, balances, pipettes
Biochemist	A lot of this you take for granted after a while in your work. You just are so comfortable with it that you don't pay too much attention to it. But it is obviously in the background of everything you do	Microscale and biochemical reactions.	Spectrometer -microlevel, flourescent microscope/ flourometer, chromatography system
Pharmaceutical chemist	Tremendously. Because there are certain rules that you have to apply when you're doing – when you're going through the developing manufacturing processes and from a quality standpoint	Laboratory scale to production scale	Mass and volume measures
Nobel chemist	I think it's very important for the understanding	Studying proteins and cells	Chemicals, filters
Nanoscale physicist	Vital	Properties that change from macroscopic to microscopic	Atomic force microscope, tunneling microscopes
Nobel physicist	n/a	Atomic scale vs. macro scale	n/a
Nobel physicist	n/a	Solar mass, light years	n/a
Biological sciences			
Cell biologist	Everything. Absolutely everything. But it's really exciting to work with all those different scales. There's actually a name for it. They don't usually let me call myself this but they call those who work at different scales but they're bigger scales, they call themselves integrated biologists	Microscopic	Microscopes, electrode tomography
Ecologist	Very important	Meshing the microscopic scale with macro scale.	Powers of 10, lab equipment
Animal behavior	It's essential	Macro to molecular scale	Maps
Entomologist	I think it's (scale) tremendously important	Surface area to volume applications with wing span	Microscopes, stage micrometer
Forester	On a scale from 1 to 10 . . . I would say a 9	Calculating board feet	Photogrammetry maps, Biltmore sticks, diameter tape, electronic mapping programs
Toxicologist	Extremely!! It's a pretty big part of it. It has now become second nature. I do it so often that I don't even think of it any longer. I don't even say, "oh I'm working with scale now". I just do it	Baseline scales of what is normal or abnormal in histopathology	Microscopes, analytical balances, graphs

Ornithologist	Like I said primary is fundamental in ecology. It's all about scale	Estimating populations. Macroscopic landscapes. Spatial scales	Populations counts, decoys
Aquatic biologist	Oh it's integral	Microscopy, scale of contamination	Maps, Scrubbers and wet scrubbers/filters
Zoologist	Yes. When trying to quantify microscopic structures like surface area and length of intestines	Grams, milligrams, microns	Micrographs, microscopes, image analysis software, electron microscopes
Neurologist	I think it would be impossible for me to practice without that concept	Comparing microscopic to macroscopic views (of a tumor)	MRI and CAT scans
Epidemiologist	I mean obviously it is very important. It comes in at so many different levels, each one of them have scales within it, scales not just the fact that I am looking at something small, its all the measurements have scales in them and at every turn you are making a judgment of if this is the best scale	Microscale to macroscale	Microscopes, computers
Earth/environmental sciences			
Archeologist	It's absolutely critical because of the very fact that we shift scale so often that you have to clearly be able to understand and define the scale relationships between the different levels in your data set	Mapping, eight inch scale, grain size, time scales	Microscopes, tape measures, optical surveyor's transit, GPS, maps, carbon dating
Geologist	Yes, it's important	Linear scale/units of measurement	Gerber scale
Meteorologist	Very important	Synoptic scale	Computer software to determine different scales
Astronomer	Oh well, everything has to do with scale. It's a little challenging as an astronomer because you can't really visualize these huge scales. You have all sorts of tricks for kind of dealing with it	Giga, parsec scales/distances, wavelengths, magnetic scales	Telescopes
Astronomer	It is probably the most important thing I think we need to understand	Brightness scale, logarithmic scale; power scales, decibel scale, logarithmic scale	Telescopes
Paleontologist	Scale is everything	Mass, linear scale SA to volume, animal body mass, fractals	X-ray technology
Engineering			
Engineer	It's fundamental in that if I didn't understand scale I would have a lot of problems figuring out would a heating/air system fit in a space or could the lighting work in that space or would the plumbing lines even go in	Standard heating/air scale, lighting scale, metric and English scales, engineer scales, architect scales	Tape measure, folding scale, scale drawings
Materials science	I think anytime you are dealing with numbers you are dealing with scale	Spatial scale, mass, nanoscale	Microscopy, lithography
Electrical engineer	I probably wouldn't be able to do it if I didn't understand scale. Because things happen on a small scale like time wise and then there's some things that happen on a large scale time wise	A million times a second or nanometers in width, very, very small scale	Reference points (150 MHz, 100 kHz)
Computer scientist	Very important	Time scale, linear scale, voltage scale	Proportions in reference to scale

(Continued)

Table 1
(Continued)

Domain	Importance of scale	Applications and types of scale	Tools
Computer scientist	n/a	Graphic scale	Computers
Traffic engineer	It's very important to my job because if I'm not accurate with my scale, or the information I get isn't accurate, it can completely throw off all of my results	Road construction plans	Computer programs, GIS, satellite and aerial photos
Other			
Chef	On a scale from 1 to 10 . . . 9	Conversions of amounts for different numbers of people	Units of measurement—liquid and mass
GIS	n/a	Changes in scale	Maps, satellite imagery
Landscaper	n/a	2D to 3D transformations, linear distance	Biltmore sticks
Cartographer	I'd say it's critical	Engineering scale	Computers, GIS
Photogrammetry	Scale is somewhat mission critical	Metric scale	Computers, maps, aerial photography
Surveyor	Extremely important	Linear scale, metric and English scales	Using body rulers to estimate height or distances, GIS systems, aerial photos
Mathematical modeling	I think it's critical to understand scale. It's absolutely one of the first things that we do. We have to know what scales we are talking about for a problem. Or differently what's the range of scales that we're interested in. I think it's just fundamental. Understanding scale is the first step	Mathematical methods for eliminating or processing the effects of small scales	Reynolds flow, density of water, kinetic viscosity of water
Muralist	n/a	I do not use grid work to make drawings. I use chalk so that if scale is not right, I can fix it	Chalk, paint, size of space
Artist	n/a	Perception in drawings	Different kinds of paint, perspectives
Automobile body mechanic	I cannot do what I do in the body shop without it	Density, volume, mass, length	Measuring tape, lift machine, frame machine, volume measurement, body measurements
Pilot	It's hypercritical	There's all kinds of scaling with electrical issues on board aircraft. (amount of fuel for large aircraft compared to smaller aircraft.)	Maps, GPS
Sculptor	It's absolutely required in my work because most of my things are somehow experienced for understanding purposes – why people interact with them. It's not like I'm telling a story and you can imagine some thing at different scales	Sawhorses at different scales. One was ninth scale, one was third scale and one was full scale	Proportions, smaller models, measuring, drawings

level 2 codes across all responses and all transcripts were read again for subcodes. The third level of coding included themes that cut across and within codes and questions. The broad themes (and subcodes) of the level three codes included: *types of scale* (types and tools); *scaling strategies* (body rulers, estimation, measurement, and anchor points); *experiences* (in-school experiences—physical experiences, models, maps, magnitude, and perspectives, and out-of-school experiences—maps, measurement, models, metric conversions, and macro/micro-applications); *thinking about scale* (cognition, automaticity, visualization, 2D/3D); and *importance of scale* (multiple scales and accuracy).

Four researchers were trained to code for the level-three codes and a random selection of 10% of the interviews were selected and coded by two teams of two researchers. The interrater agreement on the coding of the subsample of interviews was 85% and 84%. For the remaining interviews, 2 researchers coded each transcript for the level-three codes. Following the level-three coding, the transcripts were re-read for details and elaborations of patterns and themes. For example, the tools used were grouped by types of measurements being made (linear scale, mass, volume, etc.).

Assumptions and Limitations

The results that are reported in the sections that follow are specific to these participants and do not represent either the general population or others who may use scale and scaling in their jobs. These individuals were sought because they use scale in their work and as a result were likely to have had childhood experiences that contributed to their having scale-related interests and careers.

One of the limitations of the study is the assumption that participants are able to remember the events that contributed to the development of their knowledge and skills in using scale. It is highly probable that participants remember some but not all of the events that shaped their understandings of scale. There is evidence that those with extensive experience and skills in specific domains tend to organize and recall knowledge in ‘‘chunks’’ (Reif & Heller, 1982) and it is likely that this organizational process could alter the recall of developmental experiences. Furthermore, it is likely that those events that are unusually significant or meaningful may be remembered while those more mundane experiences that also contributed to understandings may have been forgotten (Brewer, 1994).

The participants in the study were all experienced and were considered to have achieved some level of success by individuals in their field. Clearly, not all of the participants would be experts. However, the sample included a range of abilities and skills in different domains of work including a subset of the participants who would be considered to be at the highest level of expertise in their field (Nobel Laureates and members of the National Academy of Sciences). It should also be acknowledged that these participants may have had an unusual interest or ability in learning about scale and may not represent either their profession or people in general. Results should be interpreted as applying to individuals with a range of knowledge and levels of experience.

Results

Uses and Types of Scales

The analysis of the transcripts revealed that a wide array of scales and applications of scales are used by the 50 participants from different fields. Table 1 shows the different occupations, applications and types of scales, scale-related tools, and models of scales used by the participants. The most common types of scales reported by the participants were those used to make linear measurements across a range of sizes from light years at the cosmic scale to nanometer measurements at the very small scale. Computer imaging associated with geographic information systems (GIS) were used by landscapers, engineers, cartographers and others whose jobs involved mapping large out door areas.

Bench scientists tended to use linear measurements as well as measurements for mass and volume. Many of the professions reportedly used unique scales such as brightness scales for astronomy, architectural scales for designing buildings, synoptic scales for meteorological measurements, and Gerber scales for interpolating and scaling plotted data.

Paralleling the use of different scales was the broad array of tools that were used in making measurements and converting scales. Microscopes and telescopes were commonly used tools, as well as a wide range of computer technologies such as computer-assisted design (CAD). Radiologic-related tools included X-rays, computerized axial tomography (CAT), and magnetic resonance imaging (MRI). As noted above, GIS technologies were also used by many of the participants.

Looking across the different professions there were commonalities in the ways that individuals reported approaching estimation and measurement tasks, as well how they conceptualized size and scale. In the sections that follow these commonalities are described.

Scaling Strategies

Body Rulers. When estimating size, many of the participants recalled using their bodies as rough measurement tools. Nineteen of the participants indicated that while working they often made quick estimates by using their bodies as a ruler. The archeologist described body measures as, ‘‘the body part is a working thing that you use just to help guide you . . . it just lets you guesstimate things.’’ The most common estimation participants made with the body was measuring linear distances through pacing. Pacing was applied in a variety of work contexts as described by a builder, landscaper, entomologist, architect, computer scientist, geologist, photogrammeter, and animal behaviorist. The use of the body as a measuring tool appeared to begin early as noted by a physicist, ‘‘ever since I was a kid I used my body length as a mental (measuring stick) . . . I wonder how big this room is. Well, if I laid down I'd probably be about that big and I am six feet three (inches).’’ A chemist recalled, ‘‘I remember learning that the creases between my fingers were approximately an inch long. This was helpful in measuring.’’

The architect in the study elaborated at length about the use of pacing as well as other body lengths as central tools to designing buildings. She explained that functional rooms are built around the space required for movements such as the space needed for doors to open or distances for chairs to move away from a table. She described how she commonly used strides and arm lengths to measure distances and eye-level to measure heights.

Other body measures mentioned were specific to particular applications within an occupation. For example, the forester described units of chains that were 66 feet long. He noted, ‘‘we know what our pace is for a chain.’’ For him, the chain was a specific unit of measure that he could estimate through pacing. Others such as the archeologist used hand spans, ‘‘a hand span for me is eight inches . . . one of my thumbs is about an inch long.’’ The neurologist noted he used his finger to estimate distance during surgery. ‘‘I remember (a specific location on the brain) . . . 6 cm up and 4 cm to the side, so for me it is 3 finger widths up and 2 over.’’ The artist described using the eye width as a unit of measurement to draw faces. She described the face as being ‘‘five eyes across’’ with ‘‘one and a half eye widths down from the eyeball to the corner of the mouth.’’

In most cases the body measure was a tool for estimating English or metric lengths such as meters, feet, centimeters, or millimeters. In other cases the body ruler was a unique measurement of size that had meaning in the context of the work domain such as the artist's eye width to estimate facial proportions.

Anchor Points. In addition to body measures, participants tended to have one or more anchor points or size references that they used routinely in their work. Eighty-three anchor points were mentioned as being used by 40 participants. These anchor points served critical roles as quick mental benchmarks that composed the individual's sense of size and scale. Anchor points were rooted in repeated experiences and in many cases were analogies or models for formal English or metric measurements.

Some anchor points were objects that were associated with very specific sizes. For example, the chemist and the zoologist both noted that they knew the size of a micron because the red blood cell is 7 µm across. The chemist noted that he conceptualizes nanometers from his work with proteins that are 100 nm in diameter. The cell biologist cited the plant embryo as an anchor point, ‘‘A plant embryo (is a reference) because that is what I live to see. It's a millimeter long and it's about 0.1–2.5 mm wide.’’ For the materials scientist, the nanometer was associated with a virus, the micron was associated with a cell (5–10 µm), and two angstroms was the size of an atom. The mathematical modeler noted that in his work everything is compared to water. ‘‘We typically work with water so we know the density of water . . .we know quite basically we are talking about 1.0 g per cubic centimeter. We know the kinematic viscosity of water and those units are 0.01 g per cubic centimeter. We know the thermal properties of water. Water is a pretty standard thing that sets our understanding of basic units.’’ Across these examples the widely used objects became the standard for other measurements.

In some cases the anchor point object did not stand as a reference for a formal measurement but instead stood for another object reference. This was seen in the work described by the paleontologist who noted, ‘‘I have two reference points. One is human size because humans relate very easily to human size. For dinosaurs I use elephants (as a reference). So I would say this dinosaur weighed five elephants.’’

For others the anchor point did not refer to a specific object but instead stood for other variables such as degree of risk. For the ecologist, cancer risk levels were a specific index such as the clean air act designating a risk level of one in a million for cancer. In the case of the meteorologist, new hurricanes were measured against one of the worst hurricanes she had experienced first hand, Hurricane Bonnie. For her, Hurricane Bonnie was a standard measure that she compared all new hurricanes against.

Experiences Learning Scale

Out-of-School Experiences. Virtually all of the participants described out-of-school experiences that they had growing up that contributed to their sense of scale. The overwhelming characteristic associated with named out-of-school experiences was that they tended to involve physical movement and activity.

Seventy-six percent of the participants attributed physical experiences such as moving through the environment by car, walking, bicycling, boating, or flying in an airplane as contributing to the development of a sense of scale. Many participants recalled the physical act of moving around as helping them learn about size and scale. Walking and bicycling though the country were mentioned as altering a participant’s sense of distance and scale. For others the movement was associated with looking at how the landscape changed as you move closer and further away. ‘‘When I was a kid I was a biker and would notice (scale). We would ride our bikes to the beach and I’d look at a tree in the distance and see how it changed as I got closer to it. Or when we would be on the beach I’d look at a boat or one of the cruise ships coming in and see how it would change. You know, I learned scale by paying attention to my surroundings’’ (muralist). One of the participants who worked with satellite imagery illustrated the power of air travel on developing a perspective of scale through a comment he had heard a child make when flying; the child was looking out of the window of the plane and asked, ‘‘when do we get small’’?

Building models and other structures emerged as the most frequently cited experience and was named by 30% of the participants. The astronomer described his love for building models as a child, ‘‘I have always had an interest in flight and flying . . .I have a hobby of collecting different scale models of airplanes. As a child, you build this model. You say ok, if it has a 1:28 scale or 1:25 scale, what does that mean? How big is that in reality? When you have a car everyone knows how big a car is, but when you have a model airplane is it a small engine plane or a big giant bomber? They tend to sell these kits with a sense of the same scale, so if I had seen the true aircraft in person, I would have had that knowledge. But, I (tried) to figure out how really big (one airplane model) is compared to a second model since they were the same size on the shelf.’’ Participants remembered building model airplanes and trains as important to learning about scale. Others recalled working with their fathers in building houses and other structures around the home.

In some cases participants’ understandings of scale were associated with making quiet observations of large-scale things such as an exhibit in a museum, a large monument, or the night sky. A paleontologist reflected, ‘‘When I was a kid in the summertime in the mountains of Oregon, the stars were beautiful . . . I cobbled myself an elevated platform bed 8 or 10 feet off the ground. I would look at the stars at night, I would pretend I couldn’t see any of the mountains or trees around me, that I was hanging loose in space. I tried to think of how different it was . . . in space.’’

Studying, reading, and creating maps were cited by 10% of the participants. This subset of the participants had a passion for maps and recalled enjoying studying maps during travel as well as creating maps. The archeologist stated, ‘‘as a kid I can remember looking at road maps whenever we would take a family trip . . . I just remember I was always interested in where I was in the landscape geographically and then figuring out how the map piece of paper related to the landscape out there.’’

School Experiences. Unlike out-of-school experiences that tended to involve physical movement, the most commonly cited in-school experiences that contributed to a sense of scale tended to be related to tools of measurement and science (14%), creating scaled maps and models (20%), and conversions of units (18%).

Many of the participants recalled doing metric and English conversions as well as making conversions from large to small scales. Microscopes and telescopes were also mentioned with enthusiasm by a number of individuals, ‘‘I loved looking under the microscope (cell biologist).’’ The astronomer recalled, ‘‘one of my first experiences was a telescope ... You see this distant little point of light in the sky, you look in the eye piece and all of a sudden you see it is actually a physical planet.’’

As was noted for out-of-school experiences, participants recalled making models and maps (the solar system or a continent) at different scales as part of formal school experiences from elementary to graduate school. In elementary school participants remembered drawing maps to scale. For example, a biologist recalled in sixth grade, ‘‘we would take a different chunk of the world . . . another student and I would paint a scale map on the board and transfer the scale by the little block copy method from a book map.’’

As students advanced through their schooling, there were more specialized experiences such as learning to make architectural drawings, making landscape designs, and mapping areas from different perspectives. There were also more experiences that required students to consider scaling effects. The mathematician recalled, ‘‘In graduate school and in college I began to be exposed to how certain things behaved the same way on different scales provided you maintain certain mathematical constraints. Certain dynamic scaling would suggest that if the fundamental processes were unchanged, you should see the same thing . . . in fluid dynamics courses there were pretty strong experiences where you could understand how you could design an airplane by building something in a small room and why that would have implications about how the plane would fly in the sky.’’

Thinking About Scale

By looking at experiences from childhood to adulthood, we sought to document how participants developed a sense of scale. The analysis of the level 3 transcript codes showed participants’ knowledge of scale was not a result of simple accretion of information but instead was tied to rapid and repeated use as well as the development of visual spatial skills.

Automaticity. As participants reflected on learning about scale in and out of school they discussed how using and changing scale had become automatic. One participant noted, ‘‘I guess it’s just one of those things that you develop without even knowing you are developing.’’ The aquatic biologist said, ‘‘I’ve got the relationships in my head about if its 350 mm that’s about 14 inches and I just automatically know that.’’ A zoologist reflected, ‘‘I used to take out my pencil and paper . . . but (now) that comes automatically.’’ Several of the participants described using scale as ‘‘second nature’’ as seen in this comment by a toxicologist ‘‘. . . now it’s second nature. I do it without even knowing it. It’s just common nature now.’’

Visualization. When applying scale, participants described a process of mentally visualizing their work at different scales. The cell biologist described his thinking this way, ‘‘I really think that (it is important to) be able to create the visual picture in your mind of the things that you’re studying. For the kind of biologist that I am, that’s absolutely the most essential thing that you can do.’’ He went on to say later in the interview, ‘‘I have a pretty good mental picture of what I think those cells are doing. And then I can scale that picture down and you can zoom in.’’ The aquatic biologist described his application of different scales in this way, ‘‘One acre is 0.42 ha. It is just imprinted on my mind. I have a vision of what size that is in my head.’’ Other participants described themselves as visual thinkers when asked about how they learned and applied scale. For example, the forester said, ‘‘I’m a visual person.’’ The mechanical engineer believed his skills in visualization were helpful for using macroscale but were a hindrance for very small scales like nanoscale, ‘‘I am totally a visual person. It’s easy for me to conceptualize what a millimeter is, a centimeter, meter, mile, or quarter mile. It’s very hard for me to conceptualize what a nanometer is. Because I can’t see it. A thousand bucky balls on the head of a pin—I don’t get that. I can’t conceptualize that.’’ One of the nationally recognized physicists specifically noted that he had to visualize the pulsars he was researching and that the mathematics alone was not enough for him to make conceptual breakthroughs in understanding phenomena at a tiny scale.

Many of the participants described having to move back and forth between two-dimensional (2D) drawings and a three-dimensional (3D) world. This skill was described as important in a variety of different domains including engineering, archeology, cell biology, construction, landscaping, ornithology, architecture, computer science, geology, traffic engineering, art, and neurology. The neurologist described how he has to look at a CAT scan and then translate it to a point of entry for introducing a shunt catheter. ''For instance if I have to place a shunt catheter tube into the ventricle place in the brain. Sometimes I want to pick a good entry point. So I will look at a slice on a CAT scan and say 'how do I translate that into the same location on the patient's head'? (We have to) co-register the 3D space on a CAT/MRI scan with the 3D space on the patient.'' The archeologist emphasized the critical nature of this 2D and 3D transformational process in recording the location of artifacts he finds, ''it's critically important that we document the three dimensional relationship of all these artifacts to each other as well as their relationship to different soil layers.''

Importance of Scale

When asked to indicate how important scale was to their work, across the participants virtually all felt like scale was important. For example, responses included, ''I can't operate without a sense of scale,'' ''scale is an integral part of what I do,'' scale is ''extremely important,'' and ''I think it would be impossible for me to practice without the concept of scale.'' Reasons given for the importance of scale included the need for accurate scale measures that can be used to make predictions such as the size of a heating and air conditioning system needed for a building or the size of plants in a landscaping plan. The need for accurate scales was cited as critical to interpreting results and communicating data to other people. For example, the ecologist stressed the need for the public to understand risk and the likelihood of a given disease.

Scale was also seen as critical when work required an individual to operate at multiple scales. In areas of science some processes and phenomena may operate differently at different scales. The nanoscale physicist noted, ''scale ends up being so important because the fact of the matter is that there are lots of physical and chemical properties and phenomenon that don't scale.'' In other cases participants noted the need to be aware of things happening on a very small scale while interpreting macroscale properties and behavior. This was cited by several of the participants such as the epidemiologist, embryologist, and cell biologist who study genetic differences that give rise to human-scale observable characteristics and properties.

Discussion

There are patterns that emerge across the data as we seek to document the trajectories of learning that occurs as individuals learn about measurement, scale, and scaling effects. The results from this study as well as research published previously in other studies suggest that scale includes both declarative and process (skills-based) forms of knowledge. Students must learn labels for types of scales, concepts of quantity, as well as a myriad of scaling strategies involved in measuring, imaging, estimating, and using measurement tools (Tretter, Jones, Andre, et al., 2006). Once a student learns common measurements such as centimeters or inches, they must move flexibly from words to symbols, to conceptual units. As the participants in this study shared, physical experiences were key in developing skills in using measurements and applying scale. Experiences such as building a dog house or a model airplane were remembered as significant events in learning scale.

Out-of-school experiences evoked more memories than in-school experiences. Was this because learning out of school involved more physical activity or perhaps these experiences were associated with other pleasant memories such as working along side of a parent or friend? In-school experiences can be routine with an intentional design of building knowledge and skills one at a time. As a consequence, it is possible that memories of these experiences were not as vividly held by participants. We cannot discount the significance of school-based experiences even if they were not remembered as vividly as those that took place out of school.

Repeated in and out-of-school experiences led to more advanced skills in using scale. Participants recalled that as they developed experience in using scale, their skills with scale became more fluid and rapid. This type of automaticity has been associated with the development of increasing expertise in a variety of different domains (Chase & Simon, 1973). These participants utilized tools such as body rulers to facilitate rapid estimations. Extensive experiences in using their bodies as rulers allowed them to have confidence in the reliability of the rough measurements that they made. Conceptual anchors like the size of an atom or the density of water were also used by participants as they made rapid conversions from one scale to another. There was not a single set of universally used anchor points, but across the different professions most participants utilized some type of anchor points in their everyday work. These conceptual anchors gave individuals meaningful points of reference that they could use while they navigated across large differences in scales.

Figure 1 shows a hypothetical model indicating a possible gradient of scale concepts and processes that develops as an individual moves from a novice through degrees of increasing experience. The different components of this model are not intended to be viewed as entirely discrete; instead these are overlapping, fluid components of scale that develop through time. The evidence for this model is pulled from previous studies as well as from the information gained from these 50 experienced professionals. The model is built on a developmental framework that includes the development of the ability to move from concrete to abstract concepts (Chin & Seigler, 2000; Siegler 1998). As children we begin to develop understandings of scale through exploring ideas of less and more as well as comparing objects and sizes as we develop a sense of numbers and units (Greeno, 1991; Lamon, 1993). From early in elementary years, we are taught skills in using measurement tools such as rulers, scales, and measuring cups (Hofstein & Lunetta, 1982). This focus on measurement tools continues throughout formal schooling and is reinforced in informal experiences such as building models and creating maps. As students move into middle school, their development of a multiplicative conceptual field (Hiebert & Behr, 1988) and their increasing ability to use proportional reasoning skills allows them to consider how changing scales influences different dimensions of materials and objects (Flavell, Miller, & Miller, 2002; Lamon, 1993; Taylor & Jones, in press). This includes being able to conceptualize the relationship between surface area to volume and understanding how this relationship can place limits on the sizes of living and nonliving things (e.g., Tourniare & Pulos, 1985; Vergnaud, 1983). Learning how to convert measurements from one system to another, or one scale to another, takes place across schooling from elementary to graduate school (Tretter, Jones, Andre, et al., 2006). Students develop increasing accuracy in making estimations as well as precise measurements. Learning how to visualize and mentally manipulate scales is a process that seems to begin in the middle years and continues with increasing experience (Tretter et al., 2006b). As shared by the participants in this study, later in schooling students learn how to create and reliably use new, unknown scales for phenomena. As seen in the interviews with these participants, with experience individuals are able to use and apply scale with increasing speed and automaticity of thought. At this end of the continuum, experienced professionals report that they can make accurate estimations without formal tools by using their bodies as rulers as well as using conceptual benchmarks to mentally move from one scale to another.

Across professions the participants emphasized the critical role that scale plays in their work. Scale was not just important, but in many cases was viewed as central to accomplishing the work-related tasks. Given the importance that these professionals placed on scale, it isn't clear whether or not educators place sufficient emphasis on teaching scale. In a previous study teachers were asked where they taught scale in their science curriculum and their responses indicated that they primarily only teach about linear size and distances (Jones et al., 2008). The K-12 science curriculum typically includes scale but concepts are often embedded in other topics (such as teaching about surface area to volume as part of a study of cells). Teachers may teach basic measurement skills but the results of this study suggests that to develop increasingly sophisticated understandings of scale, teachers may need to place more emphasis on visualizing scale, learning how to move from one scale to another, as well as creating new scales. It appears that there may be a gap between the critical role of scale as described by these participants and the current place of scale in science instruction.

Summary

The retrospective view of learning that these professionals shared showed that virtually all used scale consistently and frequently in their work. In most cases, using scale was considered to be a critical skill. Valuable in-school experiences related to scale included learning about different scales, how to covert from one scale to another, and learning to use tools such as microscopes and telescopes. Both in and out-of-school participants recalled the importance of mapping and building different things as helping them understand size and scale. Outside of the classroom, experiences that involved physical movement were recalled as significant in confronting and applying different scales. A number of hobbies such as looking at stars or putting together model airplanes were remembered as valuable in learning scale.

Experienced

- Automaticity and accuracy (Tretter, Jones, & Minogue, 2006)
- Creating reliable scales
- Relating one scale to another (Jones, Taylor, Minogue, Broadwell, Wiebe, and Carter, 2007)
- Developing accuracy in using scale (Jones, Tretter, Taylor, & Oppewal, 2008; Tretter, Jones, Andre, Negishi, & Minogue, 2006)
- Applying conceptual anchors when estimating scale

Developing

- Converting measurements and scales
- Surface area to volume relationships (Taylor & Jones, in press; Taylor & Jones, 2008)
- Being aware of changing scales (Tretter, Jones, & Minogue, 2006)
- Using body rulers for measurement and estimation
- Visualizing scales (Jones, Taylor, & Broadwell, in press; Jones, Minogue, Oppewal, Cook, & Broadwell, 2006)
- Understanding different types of scales (Tretter, Jones, & Minogue, 2006)
- Development of proportional reasoning; Visual spatial skills (Taylor & Jones, 2008)

Novice

- Developing measurement estimation skills
- Conceptualizing relative sizes (Jones & Rua, 2006; Tretter et al., 2006)
- Using measurement tools skillfully (Hofstein & Lunetta, 1982)
- Development of number sense (Greeno, 1991)

Figure 1. Trajectory of scale concept development.

Anchor points proved to be valuable in helping professionals move from one scale to another in their work. Anchor points were viewed as conceptual indexes that developed from experience and were frequently domain specific. Individuals reported that anchor points (such as the size of a blood cell) were conceptually useful in navigating across scales. For many of the participants the body served as a rough tool in estimating sizes and distances. This was true for large distances such as measuring a stream or small sizes such as measuring centimeters during surgery.

The findings of this study provided evidence of the value of repeated opportunities to use and apply scale in and out of the classroom. As one of the major themes that cross the science domains, the role of scale in developing scientific expertise may be underestimated. If we can give students the conceptual tools and experiences that are needed to develop a strong sense of scale at an early age, then we may be able to help them make more accurate connections between the human scale that we live in and the extreme scales where significant science discoveries are being made. One of the participants whose work involved modeling biological and physical phenomena noted:

> It is critical to understand scale. It's absolutely one of the first things we do. We have to know what scales we are talking about and the range of scales we are interested in. I think it is just fundamental. Understanding scale is the first step.

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Appendix A

Scale in the Professions Interview Protocol

(1) Would you briefly describe your work?
(2) Are there any childhood experiences that you can remember that helped you understand the sizes of things and scale?
(3) Can you think of examples from your experiences in school that have helped you understand scale?
(4) Can you think of examples of experiences out of school that have contributed to your sense of scale (such as hobbies)?
(5) How do you apply scale in your work?
(6) How important do you consider scale to your understanding of your work?
(7) What parts of your field (or job) would be hard to understand without an understanding of scale?
(8) Are there things you use as references or anchors when you think about sizes of different things?
(9) When you think about your work—is there a point where changes in scale make you think differently about what you do?
(10) Is there a scale where you need different materials or techniques?
(11) Have you ever created or used a unit of scale in your work? (A ''rule of thumb'' scale estimation or a rough measure or unit of estimation?)
(12) Consider different scales for: time, mass, volume, linear size (or distance). Do you use any of these different scales in your work—in what ways?
(13) Have there been Eureka or insightful moments or events that have significantly influenced your work?
(14) What experiences in school led to your successes in your work?
(15) What hobbies did you have growing up?

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